Simultaneous Analysis and Design for Eigenvalue Maximization

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A simultaneous analysis and design approach to the maximization of buckling or vibration eigenvalues is presented. Both unimodal and bimodal optimum solutions are considered. A discretization of the structure and response is used to obtain sets of nonlinear algebraic equations, which are solved numerically. The formulation is applied to the solution of the optimum design of a column supported by an elastic foundation for maximum buckling load. Two cases are considered: the optimum design of the column for a given foundation and the optimum design of the foundation for a given column. Results are compared to published solutions when possible.

I. Introduction

IN its early days, structural optimization employed the calculus of variations to obtain the Euler-Lagrange optimality differential equations, and these were solved simultaneously with the differential equations of structural response. For built-up structures modeled by finite elements, a nested approach is more typical. Resizing rules based on optimality criteria require that the structural response be calculated repeatedly for each set of trial structural design variables. This preference for the nested over the simultaneous approach is probably due to the simplicity of the structural resizing rules that are possible when the structural response is known. This simplicity contrasts with the difficulty of solving the large systems of nonlinear algebraic equations that are obtained from a simultaneous formulation.

Direct search methods, which have been gaining popularity, are commonly used in a nested approach, with the structural analysis equations repeatedly solved during each design iteration. Part of the reason for the popularity of the nested approach is that the structural analysis equations are solved by techniques quite different from those used for the design optimization.

In the late 1960's, Schmit, Fox, and co-workers[1-3] tried to integrate structural analysis and design by employing conjugate gradient (CG) minimization techniques for solving linear structural analysis problems. They found that CG methods were not competitive with the traditional direct Gaussian elimination techniques.

Recent advances in methods for solving nonlinear equations and ill-conditioned optimization problems are prompting a reassessment of the simultaneous analysis and design approach. References 4 and 5 report good experience with direct search optimization for design subject to stress and displacement constraints using a preconditioned conjugate gradient method.[6] However, as reported in Ref. 7, the direct approach does not provide a good formulation for simultaneous analysis and design subject to eigenvalue constraints. Thus, for such problems, it is necessary to look to the variational optimality methods.

One objective of the present work is to formulate the simultaneous analysis and design approach for eigenvalue maximization. The formulation leads to a set of nonlinear algebraic equations for the discretized structure. These problems often have bimodal solutions; that is, the optimum eigenvalue has two eigenvectors associated with it.[8] Thus, a bimodal formulation of the optimization is also given. A second objective of the present work is to apply the simultaneous formulation to the optimum design of beam columns with elastic foundations.

There have been a number of studies on the optimum design of structures with given foundations and eigenvalue constraints. Vibrating beams with frequency constraints were considered in Refs. 9 and 10, and columns with buckling load constraints were considered in Refs. 8 and 11–13. In Ref. 8, Kiusalaas presented an example of a simply supported column on a given foundation. He showed that the optimal solution could be bimodal; i.e., the lowest buckling load could be a repeated eigenvalue. This problem has recently been studied in more detail by Gajewski[12] and Plaut, Johnson, and Olhoff.[13] In Ref. 13, it was shown that bimodal solutions appear in certain ranges of foundation stiffness for columns with various boundary conditions. As discussed in Ref. 14, a nested approach to this bimodal problem can encounter numerical difficulties.

The optimum distribution of foundation stiffness for given structures subject to eigenvalue constraints was only studied in Ref. 15. The minimum natural frequency of a vibrating beam was maximized. Under special conditions, the optimal solution is bimodal.

The present work applies the simultaneous analysis and design approach to two problems: optimum column design with a given foundation and optimum design of the foundation for a given column. The buckling load is maximized. The nonlinear algebraic equations are solved by Powell's method,[16] and results are compared, when possible, to published solutions.

Computational efficiency, which is of major importance in comparing the nested and simultaneous approaches, is not addressed here. Rather, the objective of this work is to demonstrate the feasibility of the simultaneous approach for eigenvalue problems. For one-dimensional problems like the ones analyzed here, the traditional nested approach is more efficient computationally. However, for three-dimensional problems, past experience (e.g., Ref. 4) indicates that the simultaneous approach becomes competitive.

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II. Formulation

Optimization Problem

The smallest eigenvalue P of a vibration or buckling problem can be expressed by Rayleigh's quotient,

$$P = \min_y \frac{V(d,y)}{L(d,y)} \tag{1}$$

where d is a structural material distribution function, y the displacement function, $V(d,y)$ the elastic energy functional, and $L(d,y)$ a kinetic energy functional (for the vibration problem) or a work functional (for the buckling problem).

The design problem we consider here is to maximize P for a given amount of resources with some subsidiary constraints on d (such as upper or lower limits). This problem is written as

$$\max_d \max_y \frac{V(d,y)}{L(d,y)} \tag{2}$$

such that

$$H(d) = 0$$
$$g(x,d) \geq 0$$

where the functional $H(d)$ represents a resource constraint, x is the coordinate vector, and $g(x,d)$ is the subsidiary constraint.

The functionals $V(d,y)$ and $L(d,y)$ are homogeneous functionals of the same order, and so, instead of problem (2), it is permissible to require $L(d,y) = 1$ and form the following Lagrangian function:

$$P^* = V(d,y) - \eta\{L(d,y) - 1\} - \mu H(d)$$
$$- \int_x \Lambda(x)\{g(x,d) - T^2(x)\}\,\mathrm{d}x \tag{3}$$

where η and μ are Lagrange multipliers, Λ is a Lagrange-multiplier function, and T is a slack variable function.

Next, the unknown functions d, y, Λ, and T are discretized in space as

$$d = \sum_{i=1}^{M} a_i \bar{d}_i(x)$$
$$y = \sum_{i=1}^{N} b_i \bar{y}_i(x)$$
$$\Lambda = \sum_{i=1}^{P} \lambda_i \bar{\Lambda}_i(x)$$
$$T = \sum_{i=1}^{Q} t_i \bar{T}_i(x) \tag{4}$$

Also, a_i is replaced by β_i^2 to prevent the material distribution function d from having negative values. Substituting from Eqs. (4) into Eq. (3), P^* becomes a function of the unknown scalar quantities β_i, b_i, λ_i, t_i, μ, and η.

First-Order Conditions

The necessary conditions for an optimum are obtained by taking the first derivatives of P^* with respect to β_i, b_i, λ_i, t_i, μ, and η and setting them to zero. Thus, we obtain

Optimality conditions:

$$\frac{\partial V(d,y)}{\partial \beta_i} - \eta \frac{\partial L(d,y)}{\partial \beta_i} - \mu \frac{\partial H(d)}{\partial \beta_i}$$
$$- \int_x \Lambda(x) \frac{\partial g(x,d)}{\partial \beta_i}\,\mathrm{d}x = 0 \qquad \text{for } i = 1, \ldots, M \tag{5}$$

Stability conditions:

$$\frac{\partial V(d,y)}{\partial b_i} - \eta \frac{\partial L(d,y)}{\partial b_i} = 0 \qquad \text{for } i = 1, \ldots, N \tag{6}$$

Local inequality constraints (1):

$$\int_x \{g(x,d) - T^2(x)\}\bar{\Lambda}_i(x)\,\mathrm{d}x = 0 \qquad \text{for } i = 1, \ldots, P \tag{7}$$

Local inequality constraints (2):

$$\int_x \Lambda(x) T(x) \bar{T}_i(x)\,\mathrm{d}x = 0 \qquad \text{for } i = 1, \ldots, Q \tag{8}$$

Resource constraint:

$$H(d) = 0 \tag{9}$$

Normalization constraint:

$$L(d,y) = 1 \tag{10}$$

Equations (5–10) are nonlinear simultaneous equations with unknowns β_i, b_i, λ_i, t_i, μ, and η. After these equations are solved numerically, the optimum material distribution d and the displacement field y are obtained from Eqs. (4).

Check for Optimality

The first derivatives provide only a necessary condition for the optimum design, and there may be multiple solutions to these nonlinear equations. The true optimum solution must then be determined from these multiple solutions.

First, we need to check the Kuhn-Tucker conditions:

$$\lambda_i \geq 0 \qquad \text{for } i = 1,2, \ldots, P \tag{11}$$

Then, the second-order optimality conditions should be checked. The second-order conditions are given in Ref. 17 for a minimization problem. Our optimum design problem is a min-max problem in which the objective function P^* is maximized with respect to the material distribution variables β_i and minimized with respect to the displacement field variables b_i.

The second-order necessary conditions for optimality are

$$r_1^T [\nabla_\beta^2 P^*] r_1 < 0$$

for every r_1 such that

$$\nabla_\beta h_p^T r_1 = 0 \qquad \text{for } p = 1,2$$

$$\nabla_\beta g_m^T r_1 = 0 \qquad \text{for } m = 1,2, \ldots, P \tag{12}$$

for those constraints with $\lambda_m > 0$ where

$[\nabla_\beta^2 P^*] = [\partial^2 P^* / \partial\beta_i \partial\beta_j]$ for $i = 1, \ldots, M$ and $j = 1, \ldots, M$

$$h_1 = H(d)$$

$$h_2 = L(d,y) - 1$$

$$g_m = \int_x \{g(x,d) - T^2(x)\}\bar{\Lambda}_m(x)\,\mathrm{d}x$$

and

$$r_2^T [\nabla_b^2 P^*] r_2 > 0$$

for every r_2 such that

$$\nabla_b h_2^T r_2 = 0$$

where

$$[\nabla_b^2 P^*] = [\partial^2 P^*/\partial b_s \partial b_t] \qquad \text{for } s = 1, \ldots, N \text{ and } t = 1, \ldots, N \tag{13}$$

Bimodal Formulation

The preceding formulation gives only unimodal solutions (i.e., solutions having a single eigenvector associated with the eigenvalue). To seek the solutions with double eigenvectors, the problem is to be formulated assuming bimodality of solutions, or equality of the two lowest eigenvalues, P_1 and P_2. They are expressed in terms of the Rayleigh quotient,

$$P_i = \frac{V(d,y_i)}{L(d,y_i)} \qquad \text{for } i = 1,2 \tag{14}$$

where y_i are the corresponding eigenvectors.

Treating the bimodality condition as an equality constraint, $P_1 - P_2 = 0$, the augmented functional P^* is formed:

$$P^* = V(d,y_1) - \gamma\{V(d,y_1) - V(d,y_2)\} - \sum_{i=1}^{2} \eta_i\{L(d,y_i) - 1\} - \mu H(d) - \int_x \Lambda(x)\{g(x,d) - T^2(x)\}\,dx \tag{15}$$

The eigenvectors y_1 and y_2 need to be distinct, and this could be accomplished by including an orthogonality constraint in Eq. (15). However, in this paper, it is accomplished by the discretization procedure. Discretization for y in Eqs. (4) is replaced by

$$y_1 = \sum_{i=1}^{N/2} b_i \bar{y}_{1i}(x)$$
$$y_2 = \sum_{i=1}^{N/2} c_i \bar{y}_{2i}(x) \tag{16}$$

The first-order conditions, Eqs. (5–10), are replaced by

Optimality conditions:

$$\frac{\partial V(d,y_1)}{\partial \beta_i} - \gamma\left\{\frac{\partial V(d,y_1)}{\partial \beta_i} - \frac{\partial V(d,y_2)}{\partial \beta_i}\right\} - \sum_{j=1}^{2} \eta_j \frac{\partial L(d,y)}{\partial \beta_i} - \mu \frac{\partial H(d)}{\partial \beta_i} - \int_x \Lambda \frac{\partial g(x,d)}{\partial \beta_i}\,dx = 0 \quad \text{for } i = 1, \ldots, M \tag{17}$$

Stability conditions:

$$(1-\gamma)\frac{\partial V(d,y_1)}{\partial b_i} - \eta_1 \frac{\partial L(d,y_1)}{\partial b_i} = 0 \qquad \text{for } i = 1, \ldots, N/2 \tag{18a}$$

$$\gamma \frac{\partial V(d,y_2)}{\partial c_i} - \eta_2 \frac{\partial L(d,y_2)}{\partial c_i} = 0 \qquad \text{for } i = 1, \ldots, N/2 \tag{18b}$$

Local inequality constraints (1):

$$\int_x \{g(x,d) - T^2(x)\}\bar{\Lambda}_i(x)\,dx = 0 \qquad \text{for } i = 1, \ldots, P \tag{19}$$

Local inequality constraints (2):

$$\int_x \Lambda(x) T(x) \bar{T}_i(x)\,dx = 0 \qquad \text{for } i = 1, \ldots, Q \tag{20}$$

Bimodality constraint:

$$V(d,y_1) - V(d,y_2) = 0 \tag{21}$$

Resource constraint:

$$H(d) = 0 \tag{22}$$

Normalization constraints:

$$L(d,y_i) = 1 \qquad \text{for } i = 1,2 \tag{23}$$

A new notation e_i is introduced as variables that comprise b_j and c_j,

$$e_i = \{b_1, b_2, \ldots, b_{N/2}, c_1, c_2, \ldots, c_{N/2}\}^T \qquad \text{for } i = 1, \ldots, N \tag{24}$$

Then the second-order conditions [Eqs. (12) and (13)] are replaced by

$$r_1^T [\nabla_\beta^2 P^*] r_1 < 0 \text{ for every } r_1 \text{ such that}$$
$$\nabla_\beta h_p^T r_1 = 0 \qquad \text{for } p = 1, \ldots, 4$$
$$\nabla_\beta g_m^T r_1 = 0 \qquad \text{for } m = 1,2, \ldots, P \tag{25}$$

for those constraints with $\lambda_m > 0$ where

$$[\nabla_\beta^2 P^*] = \left[\frac{\partial^2 P^*}{\partial \beta_i \partial \beta_j}\right] \qquad \text{for } i = 1, \ldots, M \text{ and } j = 1, \ldots, M$$
$$h_1 = V(d,y_1) - V(d,y_2)$$
$$h_2 = H(d)$$
$$h_3 = L(d,y_1) - 1$$
$$h_4 = L(d,y_2) - 1$$
$$g_m = \int_x \{g(x,d) - T^2(x)\}\bar{\Lambda}_m(x)\,dx$$

and

$$r_2^T [\nabla_e^2 P^*] r_2 > 0 \qquad \text{for every } r_2 \text{ such that}$$
$$\nabla_e h_p^T r_2 = 0 \qquad \text{for } p = 1,2,3 \tag{26}$$

where

$$[\nabla_e^2 P^*] = \left[\frac{\partial^2 P^*}{\partial e_s \partial e_t}\right] \qquad \text{for } s = 1, \ldots, N \text{ and } t = 1, \ldots, N$$
$$h_1 = V(d,y_1) - V(d,y_2)$$
$$h_2 = L(d,y_1) - 1$$
$$h_3 = L(d,y_2) - 1$$

Computer Implementation

Because the method described requires the solution of a large system of nonlinear equations, a systematic solution process was adopted to obviate the need for an exhaustive search through the multiple solutions. The overall solution process is as follows:

1) Start with small numbers for M, the number of material variables, and N, the number of response variables.

2) Select uniform initial values for material variables and the corresponding first or second eigenvector as initial values for the response variables.

3) Obtain the solution of the first derivative equations.

4) Check the Kuhn-Tucker conditions and the second-order conditions. If satisfied, double the number of variables, M and N. If not, choose new initial values of β_i and b_i (or e_i), and go to step 3.

5) Stop when M is large enough to approximate a smooth material distribution.

For solving these nonlinear systems of equations [Eqs. (5–10) or (17–23)], an IMSL routine, ZSPOW, is used. ZSPOW is based on the MINPACK subroutine HYBRD1, which uses a modification of Powell's hybrid algorithm.[16] This algorithm is a variation of Newton's method which uses a finite-difference approximation to the Jacobian and takes precautions to avoid large step sizes or increasing residuals.

III. Optimal Column on Elastic Foundation

Unimodal Formulation

The problem considered in this section is a simply supported elastic column on an elastic foundation (see Fig. 1). A compressive axial force P is applied at the ends of the column, and the foundation stiffness K is assumed to be constant (Winkler-type foundation). In this optimization problem, the objective is to maximize the lowest buckling load while the total volume of the column remains fixed. The lowest buckling load P is expressed in terms of the Rayleigh quotient,

$$P = \min_Y \frac{\int_0^L EI(Y'')^2\,dX + \int_0^L KY^2\,dX}{\int_0^L (Y')^2\,dX} \tag{27}$$

where X is the axial coordinate, L the column length, and $Y(X)$ the transverse deflection.

For computational simplicity, the bending stiffness of the column, $EI(X)$, is assumed to be proportional to the cross-sectional area $A(X)$:

$$EI(X) = cEA(X) \tag{28}$$

where c is a constant. This is the case for a sandwich column or a column with constant depth and varying width.[13]

Introducing nondimensional quantities x, $y(x)$, $\alpha(x)$, p, and k by

$$x = \frac{X}{L}, \qquad y(x) = \frac{Y(xL)}{L}, \qquad \alpha(x) = \frac{A(xL)}{A_u}$$
$$p = \frac{PL^2}{EI_u}, \qquad k = \frac{KL^4}{EI_u} \tag{29}$$

where A_u and EI_u correspond to a uniform column with the same total volume, the nondimensional buckling load p is expressed as

$$p = \min_y \frac{\int_0^1 \alpha(y'')^2\,dx + k\int_0^1 y^2\,dx}{\int_0^1 (y')^2\,dx} \tag{30}$$

and the constraint of given total volume becomes

$$\int_0^1 \alpha\,dx = 1 \tag{31}$$

Then the augmented functional p^* is

$$p^* = \int_0^1 \alpha(y'')^2\,dx + k\int_0^1 y^2\,dx - \eta \times\left\{\int_0^1 (y')^2\,dx - 1\right\} - \mu\left\{\int_0^1 \alpha\,dx - 1\right\} \tag{32}$$

where η and μ are Lagrange multipliers.

The buckling mode is approximated as a series of sine functions that are the buckling modes for a column with a uniform cross section,

$$y = \sum_{i=1}^{N} b_i \sin(i\pi x) \tag{33}$$

where N is the number of modes.

The cross-sectional area α is assumed to be symmetric about the midspan. To represent α, M equidistant nodes are selected in the region $0 < x < \frac{1}{2}$ [the first node is at $x = 1/(2M)$ and the Mth node is at $x = M/(2M+1)$], and α is assumed to vary linearly between the nodes. Then α is expressed as a linear combination of the a_i, where a_i denotes the cross-sectional area at node i. Also, a_i is replaced by β_i^2 to prevent the cross-sectional area α from having negative values. Then the augmented functional p^* is transformed to a function that is expressed in terms of the variables β_i, b_i, μ, and η. By taking the partial derivatives of p^* with respect to these variables, the first derivative conditions [Eqs. (5–10)] and the second derivative conditions [Eqs. (12) and (13)] are obtained.

Bimodal Formulation

The two lowest buckling loads, p_1 and p_2, expressed by the Rayleigh quotient are

$$p_i = \frac{\int_0^1 \alpha(y_i'')^2\,dx + k\int_0^1 y_i^2\,dx}{\int_0^1 (y_i')^2\,dx} \qquad \text{for } i = 1,2 \tag{34}$$

where y_i are the corresponding buckling modes. The bimodality condition is treated as an equality constraint,

$$p_1 - p_2 = 0 \tag{35}$$

Normalizing the buckling modes y_i such that the denominators of the Rayleigh quotient are unity, the augmented functional p^* is constructed:

$$p^* = p_1 - \gamma(p_1 - p_2) - \sum_{i=1}^{2} \eta_i\left\{\int_0^1 (y_i')^2\,dx - 1\right\} - \mu\left\{\int_0^1 \alpha\,dx - 1\right\} \tag{36}$$

where γ, η_1, η_2, and μ are Lagrange multipliers.

Since the model treated has symmetric boundary conditions, it is expected that the buckling modes associated with the lowest buckling loads are symmetric and antisymmetric. Therefore the modes y_1 and y_2 are discretized as follows:

$$y_1 = \sum_{i=1}^{N/2} b_i \sin(2i-1)\pi x$$
$$y_2 = \sum_{i=1}^{N/2} c_i \sin(2i\pi x) \tag{37}$$

Then the first derivative conditions and the second derivative conditions are obtained from Eqs. (17–26).

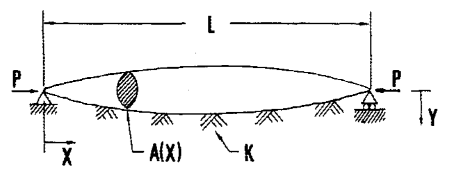

Fig. 1 Geometry of column and foundation.

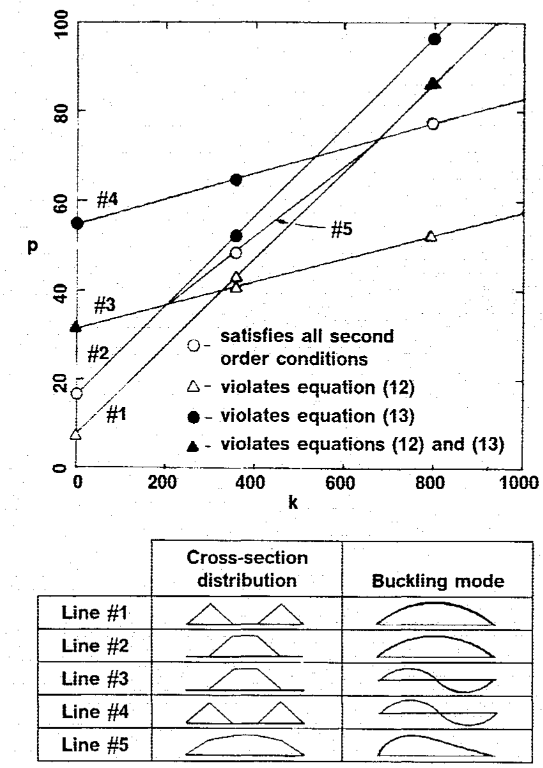

Fig. 2 Example using $M = 2$, $N = 2$.

Table 1 Buckling loads for different numbers of variables ($k = 1000$)

$M \times N$	2×5	4×10	8×20	16×40
Buckling loads (bimodal)	—	70.400	73.040	73.008
Buckling loads (unimodal)	74.999	72.719	73.051	—
Ratio of the first two buckling loads (unimodal)	0.7352	0.8182	0.9660	—

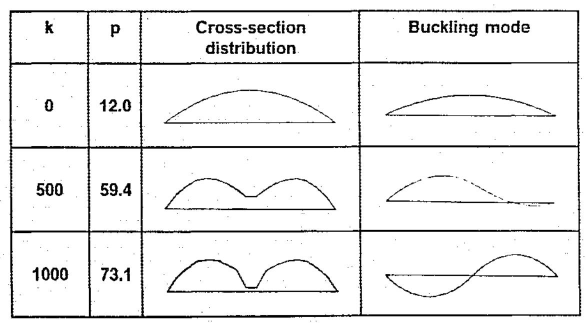

Fig. 3 Unimodal optimum designs for column with fixed foundation ($M = 8$, $N = 20$).

Results and Discussion

To show how the second-order conditions work for the minmax problem, a simple example with two cross-sectional variables and two buckling mode variables is solved first. Lines #1, #2, #3, #4, and #5 in Fig. 2 are nondimensional buckling loads for solutions that satisfy the first-order conditions. The second-order conditions are checked for three foundation stiffnesses: $k = 0$, 350, and 800.

The results of the second-order conditions demonstrate their physical interpretations. For example, there are four solutions for $k = 0$, two each for two types of cross sections. The solution on line #1 gives the minimum value of the first buckling load, and the one on line #2 gives the maximum value of the first buckling load; the solution on line #3 gives the minimum value of the second buckling load, and the one on line #4 gives the maximum value of the second buckling load. The second-order conditions obtained are in accord with these physical interpretations: Eq. (12) is violated on lines #1 and #3, indicating that the structure can be changed to increase the buckling load, and Eq. (13) is violated on lines #3 and #4, indicating that there is a lower buckling mode. Only one solution, the one on line #2, satisfies the second variation conditions when $k = 0$, and is the true optimum. In general, whenever a solution violates Eq. (12), another material distribution exists for which the lowest buckling load is higher, and when a solution violates Eq. (13), the buckling load is not the lowest one for the given material distribution.

A computer program was written to implement the method described in the previous sections. The program starts with $M = 2$ and $N = 5$ and increases them gradually to $M = 16$ and $N = 40$. Table 1 shows the dependence of the solution on the number of terms in the discretization when $k = 1000$. As can be seen in Table 1, the unimodal solution has a higher buckling load than that of the bimodal solution. This is due to the discretization process, which replaces the bimodal solution with two almost equal buckling loads. However, as shown in Table 1, the ratio of the first buckling load and the second buckling load for the unimodal formulation approaches unity, and the buckling load converges to that of the bimodal formulation as the number of variables is increased. Also, the solution of the unimodal formulation failed for the case $M = 16$ and $N = 40$. This may indicate that a solution does not exist or that convergence is prevented by the nonlinear equation solver shuttling back and forth between the two solutions.

The mode shapes and material distributions are plotted in Fig. 3 for the unimodal formulation with $M = 8$, $N = 20$, $k = 0$, 500, and 1000 and in Fig. 4 for the bimodal formulation with $M = 16$, $N = 40$, $k = 500$ and 1000. The results are compared with those obtained by Plaut, Johnson, and Olhoff[13] in Table 2 and show good agreement.

IV. Optimal Foundation for Uniform Column

Unimodal Formulation

In the previous section, we optimized columns that are attached to given foundations. We now consider the problem of determining the optimal foundation for a given uniform column. In this problem, the objective is to maximize the lowest buckling load while the total foundation stiffness remains fixed. The lowest buckling load P is given by Eq. (27). Introducing nondimensional quantities $k(x)$, p, and k_T, besides x and $y(x)$ in Eq. (29),

$$p = \frac{PL^2}{EI}, \qquad k_T = \frac{K_T L^3}{EI}, \qquad k(x) = \frac{K(xL)L^4}{EI} \tag{38}$$

where K_T is the total foundation stiffness, the nondimensional buckling load p is expressed as

$$p = \min_y \frac{\int_0^1 (y'')^2 \, dx + \int_0^1 k y^2 \, dx}{\int_0^1 (y')^2 \, dx} \tag{39}$$

Table 2 Buckling loads of optimal columns with different foundation stiffnesses

k	0	500	1000
Plaut et al.[13]	12.0	58.6	71.9
Current study	12.0	59.3	73.0

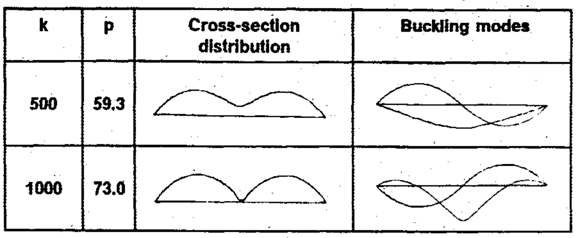

Fig. 4 Bimodal optimum designs for column with fixed foundation ($M = 16$, $N = 40$).

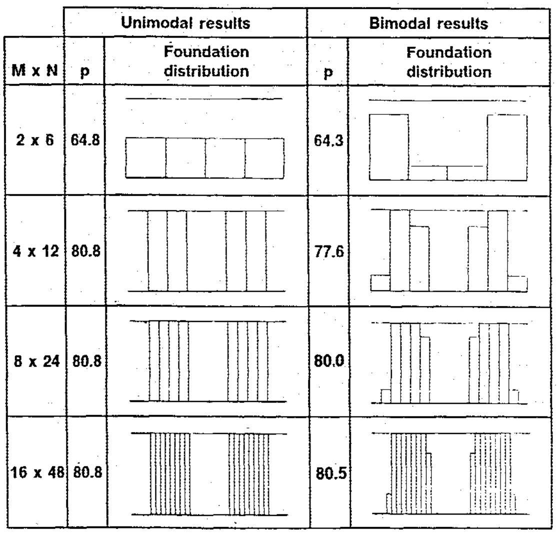

Fig. 5 Optimum foundation designs for unimodal and bimodal formulations ($k_T = 1000$, $k_{max} = 2000$).

and the constraint of given total foundation stiffness becomes

$$\int_0^1 k \, dx = k_T \tag{40}$$

Additionally, we impose the maximum foundation constraint

$$k - k_{max} \leq 0 \tag{41}$$

Then the augmented functional p^* is

$$p^* = \int_0^1 (y'')^2 \, dx + \int_0^1 ky^2 \, dx - \eta \left\{ \int_0^1 (y')^2 \, dx - 1 \right\} - \mu \left\{ \int_0^1 k \, dx - k_T \right\} - \int_0^1 \Lambda(x) \{ k + T^2(x) - k_{max} \} \, dx \tag{42}$$

where η, μ, and $\Lambda(x)$ are Lagrange multipliers and $T(x)$ is a slack variable.

The buckling mode is approximated as a series of sine functions [Eq. (33)]. The foundation distribution k, Lagrange multipliers Λ, and slack variable function T are all assumed to be symmetric about midspan. To represent these functions, M equidistant nodes are selected in the region $0 < x < \frac{1}{2}$, and k, Λ, and T are assumed to be constant between the nodes. The foundation stiffness in the ith segment is denoted β_i^2 to prevent negative values. Then, the augmented functional p^* is expressed in terms of the variables β_i, b_i, Λ_i, T_i, μ, and η. By taking the partial derivatives of p^* with respect to these variables, the first-order conditions [Eqs. (5–10)] and the second-order conditions [Eqs. (12) and (13)] are obtained.

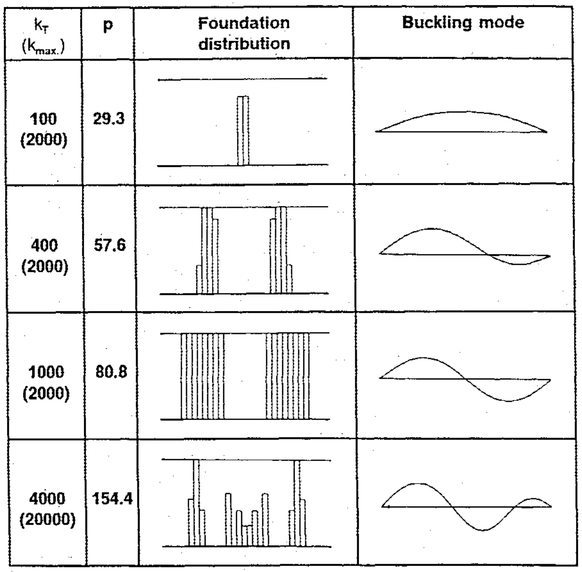

Fig. 6 Optimum foundation designs for unimodal formulation ($M = 16$, $N = 48$).

Bimodal Formulation

The two lowest buckling loads, p_1 and p_2, expressed by the Rayleigh quotient are

$$p_i = \frac{\int_0^1 (y_i'')^2 \, dx + \int_0^1 ky_i^2 \, dx}{\int_0^1 (y_i')^2 \, dx} \quad \text{for } i = 1,2 \tag{43}$$

where y_i are the corresponding buckling modes.

With the bimodality constraint [Eq. (35)], the augmented functional p^* is constructed:

$$p^* = p_1 - \gamma(p_1 - p_2) - \sum_{i=1}^{2} \eta_i \left\{ \int_0^1 (y_i')^2 \, dx - 1 \right\} - \mu \left\{ \int_0^1 k \, dx - k_T \right\} - \int_0^1 \Lambda(x) \{ k + T^2(x) - k_{max} \} \, dx \tag{44}$$

where γ, η_1, η_2, μ, and $\Lambda(x)$ are Lagrange multipliers.

The buckling modes y_1 and y_2 are discretized using Eqs. (37). Then, the first-order conditions and the second-order conditions are obtained from Eqs. (17–26).

Results and Discussion

The program starts with $M = 2$ and $N = 6$ and increases them gradually to $M = 16$ and $N = 48$. Figure 5 shows foundation distributions for various values of M and N at $k_T = 1000$ and $k_{max} = 2000$. Again, it is observed that the bimodal solutions are lower. The total CPU time (IBM 3084) when $k_t = 400$ for $M = 16$ and $N = 48$ was 39.7 s with the unimodal formulation and 36.7 s with the bimodal formulation.

For $M = 16$, $N = 48$, and several combinations of k_T and k_{max}, the optimal foundation distributions and corresponding mode shapes are plotted in Fig. 6 for the unimodal formulation

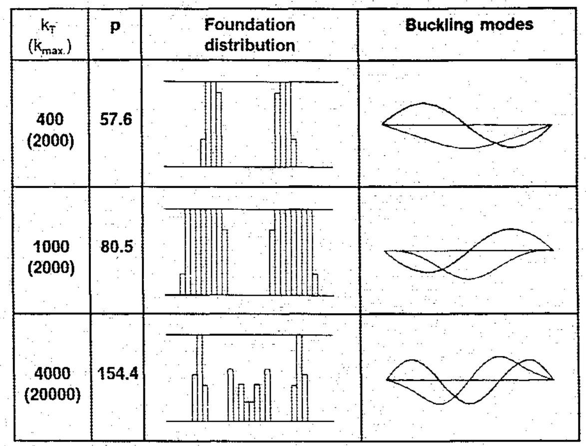

Fig. 7 Optimum foundation designs for bimodal formulation ($M = 16$, $N = 48$).

Table 3 Buckling loads for optimum and uniform foundations ($M = 16$, $N = 48$)

k_T	k_{max}	Buckling load p (unimodal results)	p_{unif}	Increase, %
100	2000	29.3	20.0	47
400	2000	57.6	49.6	16
1000	2000	80.8	64.9	25
4000	20,000	154.4	133.9	15

and in Fig. 7 for the bimodal formulation. In Fig. 6, the optimal solution tends to place foundation stiffness in regions where the buckling mode has its largest deflections. In Fig. 7, the mode shape with the highest number of maxima and minima seems to govern the placement of the foundation stiffness. For instance, if $k_T = 1000$ and $k_{max} = 2000$, there is no stiffness in the central region, where the symmetric mode has its largest deflection, and the stiffness is located about the locations of the maximum and minimum of the antisymmetric mode.

For a uniform pinned-pinned column attached to a uniform foundation, the buckling load is as follows:[18]

for the integer n such that

$$(n-1)^2 n^2 \pi^4 \leq k_T \leq n^2 (n+1)^2 \pi^4 \tag{45}$$

we have

$$p_{unif} = n^2 \pi^2 + \frac{k}{n^2 \pi^2} \tag{46}$$

The buckling loads associated with the optimal foundations are compared with those for uniform foundations in Table 3. The increase in buckling load resulting from an optimization of the foundation stiffness distribution can be substantial.

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